# Transactions Letters

# An Exact Markov Process for Multihop Connectivity via Intervehicle Communication on Parallel Roads

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***Abstract*—This paper identifies a Markov process for the multihop connectivity along two parallel roads through inter-vehicle communication. Vehicles are assumed to follow two Poisson processes on both roads. Exact mean, variance and probability distribution of the instantaneous propagation distance are derived. A closed form approximation for the expected distance is also proposed.**

***Index Terms*—Network inter-vehicle communication; vehicular ad hoc networks; Markov process.**

## I. Introduction and the Research Problem

Ad hoc vehicular networks (VANETs) with short range wireless communication have been under intensive study in recent years because of their promise to improve roadway mobility and safety. As locations of equipped vehicles traveling on highways exhibit high randomness, connectivity between vehicles is an important notion critical to the system design in terms of efficiency and reliability [1]–[6].

We study instantaneous information propagation through traffic along two parallel roads by developing exact models, different from those in the literature that are only on single road traffic [7]–[9](or treat multiple roadways as one [10], [11]). Earlier, Wang *et al.* [12] study the same problem by identifying an *approximate* Bernoulli process for traffic on two roads. This study contrasts with the literature of node connectivity on mobile ad hoc networks assuming a continuous two-dimensional space ([13], [14]).

In the study problem, two sufficiently long straight roadways $R_1$ and $R_2$ of traffic are a distance $d$ apart. Traffic densities on the two roads are $\lambda_1$ and $\lambda_2$, respectively. Vehicles follow independent Poisson processes on both roads. Starting from a vehicle $N_1$ on one road, information is propagated in a direction. Vehicles on both roads within a transmission range $L$ are able to receive and instantly further transmit the information forward. When there is no new vehicle present within range, the propagation process terminates. The propagation distance measures from the initiating vehicle to the furthest receiving one along the direction of the two roads. The objective is to characterize the distance of information propagation as a function of the vehicle densities, road separation distance and transmission range.

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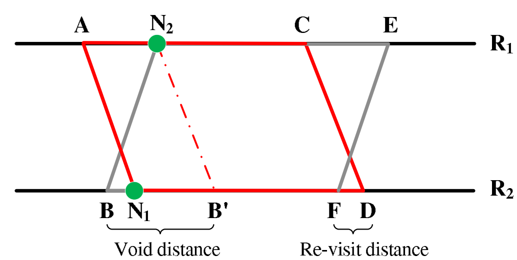



Fig. 1. Transition from type 2 to 1 region as $N_2$ receives from $N_1$.

The notions of node and vehicle are interchangeable here. The connectivity between vehicles is topologically measured by a transmission range. In addition, we only consider the case $0 \leq d \leq \frac{\sqrt{3}L}{2}$ because of technical tractability.

## II. Modeling with a Markov Process

### A. Transmission Regions

We have identified a Markov process that depends on a concept of transmission region. Take an example in which a transmitter node $N_1$ is on road $R_2$ as in Figure 1. Suppose $C$ and $D$ are the two right most nodes that $N_1$ can reach on the two roads, assuming propagation goes *rightwards*. Consider a parallelogram $N_1ACD$ where $|N_1D| = |N_1C| = |AC| = L$. We refer to $N_1ACD$ as the *transmission region* of type 2 associated with $N_1$. Similarly, a node $N_2$ present on road $R_1$ also has its transmission region $N_2BFE$, referred to as transmission region of type 1, as in Figure 1. The type of transmission region depends on whether the associated vehicle is present on $R_1$ or $R_2$. An inherent characteristic is a segment $AB'$ as in Figure 2, whose length is denoted by $r$, where $\frac{r}{2} = L - \sqrt{L^2 - d^2}$.

The case $d \leq \frac{\sqrt{3}L}{2}$ guarantees that an entire transmission region be within the transmission range of the according vehicle. If no vehicle is present in the transmission region,

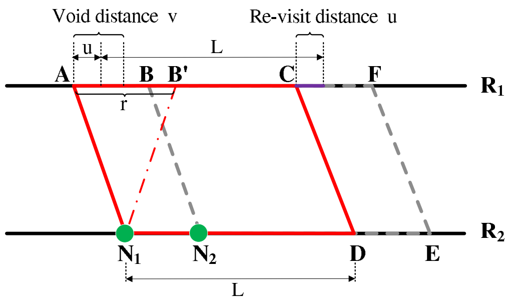



Fig. 2. Region of $N_1$ with $(v, u)$ transiting into another type 2 region.

a gap $L$ on both roads terminates the propagation. Clearly, information propagates forward from a node *only if* there is a vehicle present on the segments of the according transmission region.

### B. Transition Between Regions and State Parameters

The propagation process between the two roads is a process of transition within and between the transmission regions of the two types. In this section, we delineate the transition process and introduce two parameters associated with each transmission region: void distance and re-visit distance. A transition state is characterized by a transmission region and its two parameters.

Consider a transmitter vehicle denoted by $N_1$ on road $R_2$ with a transmission region of type 2 $N_1ACD$ as in Figure 1 and 2. We consider two cases of transition from $N_1$. In the first case as shown in Figure 2, the *first* node $N_2$ is on road $R_2$. By the term 'first' on road $R_2$, we mean no node presence in the intervals $AB$ and $N_1N_2$, and one node presence ($N_2$) whereas $N_1N_2 = AB$. Throughout the paper later, the term 'first vehicle' has the similar meaning. In this case, the transmission region of type 2 ($N_1ACD$) associated with node $N_1$ transits into a transmission region of the same type ($N_2BFE$) associated with the *first* node $N_2$. It may be equivalently taken as moving the transmission region of $N_1$ rightwards and stopping wherever the line $AN_1$ 'hits' a node on either road. In this particular case, the first node 'hit' is $N_2$ on road $R_2$. This is an example of transition between the same regions of type 2. Through this transition, the propagation distance has moved forward by $|N_1N_2|$.

The second type of transition is from a type 2 region of node $N_1$ into a region of type 1, as in Figure 1. Still consider moving the line $AN_1$ rightwards and stopping as a new line $B'N_2$ when the first node hit takes place at $N_2$ on road $R_1$. In this case, there is no node presence in $N_1B'$ and $AN_2$, but one node at location $N_2$ on road $R_1$. Now the node $N_2$ becomes the new transmitter node. Associated with node $N_2$ is a transmission region of type 1, $N_2BFE$.

Note that when transiting from a type 2 region into type 1 as shown in Figure 1, there is an associated change in two parameters. In the new type 1 region associated with $N_2$, there is a segment $BB'$ in which there is no vehicle to be considered for future propagation: those nodes in $BN_1$, if any at all, have been used prior to reaching node $N_1$. In addition, the segment $N_1B'$ is proven to be void of vehicles. Therefore, we call this segment $BB'$ as in Figure 1 *void distance*, first of the two state parameters. Ignoring nodes in the void distance does not compromise the capability of further propagation from the current transmitter. In addition, a segment $DF$ was in the previous transmission region for $N_1$, but not in this new transmission region for $N_2$. In order not to compromise the capability of propagation due to this transition from type 2 into type 1, we shall not drop this segment $DF$ from consideration. In doing so, we associate with the resulting type 1 region $N_2BFE$ a second parameter, *re-visit distance*. This means that the propagating capability of the new region of type 1 comes from its effective coverage region $B'FEN_2$ plus the re-visit distance $DF$, as seen in Figure 1. The capability of effective additional propagation can therefore be well defined by the resulting new transmission region with its two parameters. Note that any re-visit distance is due to an advance node, $L$ distance prior to the end of the re-visit distance, on the same road, as $N_1$'s being in relation to $D$ in Figure 1.

*Definition 1:* A transition state, denoted by $S_i(v, u)$, where $i = 1, 2$, is represented by the transmission region of type $i$ with two associated parameters, void distance $v$ and re-visit distance $u$.

A type 1 region transits in a similar fashion. Therefore, the process of information propagation is equivalent to a Markov process that transits between states $S_i(v, u)$,$i = 1, 2$, with each transition moving the information by a horizontal distance. In what follows, we use $D_i(v, u)$ for the random propagation distance starting from a node location with a state $S_i(v, u)$. We further let $d_i(v, u) = E[D_i(v, u)]$ and $V_i(v, u) = Var[D_i(v, u)]$ for the expectation and variance, respectively.

*Proposition 1:* There holds $u \le v \le r$ for $d \le \frac{\sqrt{3}}{2}L$.

### C. Models for the Propagation Distance

Suppose a transmitter node $N_1$ on road $R_2$ has a propagation distance $D_2(v, u)$. Take Figure 2 here for example. $D_2(v, u)$ can be recursively expressed depending on location of the next node in the transmission region of $N_1$. Three cases of first node location are identified: <u>Case 1</u>: First node on road $R_2$; <u>Case 2</u>: First node on road $R_1$; <u>Case 3</u>: No node presence in the transmission region of $N_1$ and its re-visit distance. The first vehicle location $t$, the horizontal 'move' of the left edge $AN_1$ as in Figure 2 in order to 'hit' the first vehicle, in Case 1 falls into three ranges: $[0, u]$, $(u, v]$ and $(v, L]$, resulting in states $S_2(v - t, u - t)$, $S_2(v - t, 0)$ and $S_2(0, 0)$,respectively. Similarly, $t$ falls into three ranges in Case 2: $[v, r]$, $(r, L]$ and $(L, L + u]$, resulting in states $S_1(r, r - t)$, $S_1(r, 0)$ and $S_1(r-(t-L), 0)$, respectively. In Case 3, if the last transition is one between types, the last receiving vehicle in this transition process may not be horizontally the furthest. Therefore, an adjustment to the horizontal distance might be needed. The revisit distance can be used to locate that prior vehicle and therefore can tell whether an adjustment is necessary. We develop expectation, variance and probability distribution of the propagation distance, all by conditioning on the first vehicle location. Details of the derivation are available in [15].

*Proposition 2:* The propagation distance along both roads satisfies the following relationship.

$$\begin{aligned}
&D_i(v,u) = \\
&\int_0^L \lambda_i e^{-t\lambda_i-[t-v]_+\lambda_j}\left(t+D_i^{'}([v-t]_+,[u-t]_+)\right)dt \\
&+\int_v^L \lambda_j e^{-(t-v)\lambda_j-t\lambda_i}\left(t-\frac{r}{2}+D_j^{'}(r,[r-t]_+)\right)dt \\
&+\int_L^{L+u} \lambda_j e^{-(t-v)\lambda_j-L\lambda_i}(t-\frac{r}{2}+D_j^{'}(r-(t-L),0))dt \\
&+e^{-L\lambda_i-(L+u-v)\lambda_j}\left[u-\frac{r}{2}\right]_+, \qquad (1)
\end{aligned}$$

where $D_i(v,u)$ and $D_i'(v,u)$ are i.i.d. for $i=1,2$; $[x]_+ = max\{0,x\}$. Replacing each $D_i(\cdot,\cdot)$ and $D_i^{'}(\cdot,\cdot)$ with the according $d_i(\cdot,\cdot)$ gives the expected distances, $i=1,2$.

$$\begin{aligned}
&V_i(v,u) = \\
&\int_0^L \lambda_i e^{-t\lambda_i-[t-v]_+\lambda_j}\left[(t+d_i([v-t]_+,[u-t]_+)-d_i(v,u))^2\right. \\
&\left.+V_i([v-t]_+,[u-t]_+)\right]dt \\
&+\int_v^L \lambda_j e^{-(t-v)\lambda_j-t\lambda_i}[(t-\frac{r}{2}+d_j(r,[r-t]_+)-d_i(v,u))^2 \\
&+V_j(r,[r-t]_+)]dt \\
&+\int_L^{L+u} \lambda_j e^{-(t-v)\lambda_j-L\lambda_i}[(t-\frac{r}{2}+d_j(r-(t-L),0) \\
&-d_i(v,u))^2+V_j(r-(t-L),0)]dt \\
&+e^{-L\lambda_i-(L+u-v)\lambda_j}(\left[u-\frac{r}{2}\right]_+-d_i(v,u))^2. \qquad (2)
\end{aligned}$$

where $(i,j)=(1,2)$ or $(2,1)$ (End of Proposition 2).

The variance $V_i(v,u)$ is easily derived by using the Law of Total Variance and by conditioning on the *first* vehicle location. In addition, Proposition 1 applies in Proposition 2 above.

We denote by $P_i(x,v,u)$ the probability of propagation beyond a horizontal distance $x$ starting with a state $S_i(v,u)$. Regarding probability distribution of propagation distance, we first define a function $P_i(x,v,u)=1$, if $x \le max\{0, u-\frac{r}{2}\}$, $i=1,2$. Detailed rationale is provided in [15].

*Proposition 3:* The propagation probability satisfies the following with $(i,j)=(1,2)$ or $(2,1)$.

$$\begin{aligned}
&P_i(x,v,u) = \\
&\int_0^L \lambda_i e^{-t\lambda_i-[t-v]_+\lambda_j}P_i(x-t,[v-t]_+,[u-t]_+)dt \\
&+\int_v^L \lambda_j e^{-(t-v)\lambda_j-t\lambda_i}P_j(x-t+\frac{r}{2},r,[r-t]_+)dt \qquad (3) \\
&+\int_L^{L+u} \lambda_j e^{-(t-v)\lambda_j-L\lambda_i}P_j(x-t+\frac{r}{2},r-t+L,0)dt.
\end{aligned}$$

Our additional numerical tests in [15] show a clear Gamma type of curve for $P_i(x,v,u)$ when the traffic densities are high.

## III. Closed Form Approximation with a Single Road

A meaningful question is whether there is a way to consolidate the two road densities to use as on a single road for study of propagation distance. Based on a rough observation that road $R_1$ of a length $L-\frac{r}{2}$ is horizontally right of the transmitting vehicle on road $R_2$ in a Type 2 region (refer to Figure 2), we propose the following formula when consolidating the two densities, which is equivalent to evening out the traffic in $L-\frac{r}{2}$ on road $R_1$ onto a longer distance $L$ on $R_2$, $\lambda = \lambda_1+\lambda_2-\alpha\cdot\frac{r}{2L}\cdot\lambda_1$, where $\lambda_2 \ge \lambda_1$. The value $\lambda$ is then used in the formulas for a single road as in [9]: Expectation $=\lambda^{-1}(e^{\lambda L}-L\lambda-1)$ and Variance $=\lambda^{-2}(e^{2\lambda L}-2L\lambda e^{\lambda L}-1)$.

Typically $\alpha \in [1.0, 2.0]$. Our test shows that $\alpha = 1.0$ leads to overestimates while $\alpha=2.0$ leads to underestimates of the expectation. Therefore, we propose $\alpha=1.5$. As special cases, when the road separation distance $d$ is zero, the consolidated density becomes the sum of the two.

## IV. Numerical Tests

We solve the integral equations by simply discretizing the integrations and solving the resultant arrays of equations. Because $r$ is not a large value, the number of states $(v,u)$ resulting from discretization is not too large. Analytical results are validated by simulation. Each simulation generates a snapshot of traffic and measures the propagation distance. The vertical bars in Figure 3 and 4 are one standard deviation about the corresponding mean expectation and mean variance based on 20 runs, each conducting 2000 simulations and giving an ‘expectation’ and a ‘variance’.

In the numerical tests, we scale the transmission range to be a standard unit 1.0. All other measures including the node density are scaled accordingly. In light of [16], an average density for freeway at level of service A (a stable density) is about 5.5 vehicles per mile per lane (vpmpl). When a percentage of equipped vehicles (also referred to as market penetration rate) is applied, a significantly lower equipped vehicle density could result. Additionally, we set $R_2$ to have the higher traffic density.

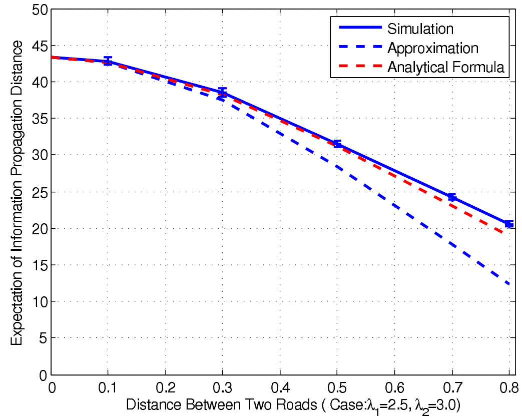
*Effect of Road Separation on Propagation Distance* As in Figure 3, road separation decreases the propagation distance at an increasing rate, especially at large traffic densities. When the road separation distance increases from zero to about 0.4 at low traffic density, the decrease of both expectation and variance is insignificant.

*Effect of Traffic Density on Propagation Distance* Figure 4(a) and 4(b) indicate that the expected propagation distance increases at a slightly increasingly rate with traffic density and that the variance of propagation distance increases with traffic density at a larger rate than the expected distance.

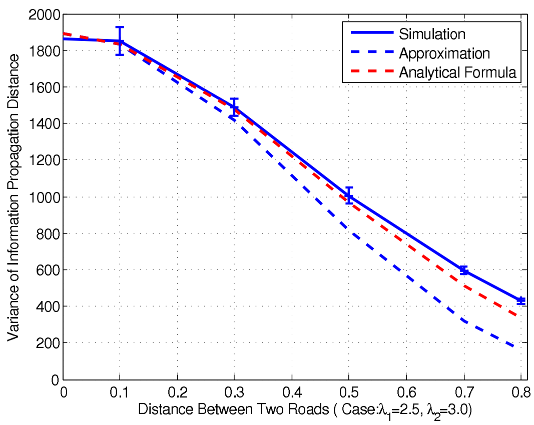
*Performance of Approximation* In terms of both the expected distance and the variance, the approximation agrees exceptionally well with the exact solution when the vehicle density is relatively low and when the road separation is small as in Figure 3.

## V. Conclusion

Interactions of nodes on the two roads clearly undermine the applicability of the connectivity models developed for a single road. We explicitly consider road separation in this paper and identify an inherent Markov process that equivalently determines the propagation distance. Exact models are proposed for the information propagation distance. This research measures

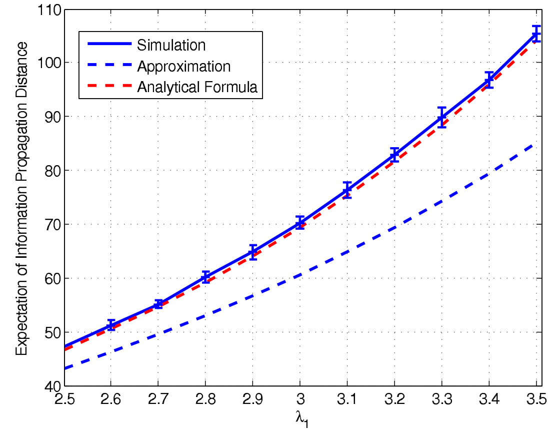



(a)

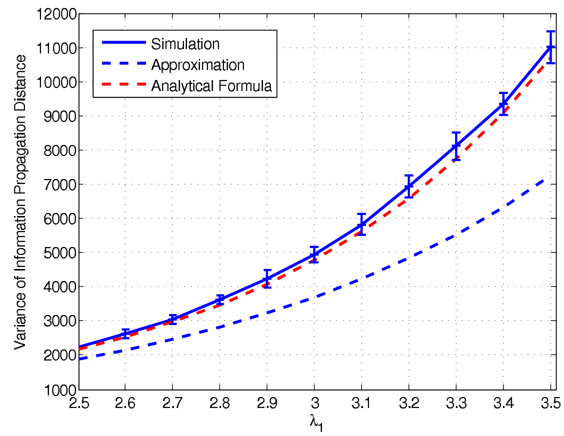



(b)

Fig. 3. Propagation distance at $\lambda_1 = 2.5, \lambda_2 = 3.0$.




(a)




(b)

Fig. 4. Effect of traffic density at $d = 0.5$ and $\lambda_2 = 3.5$.

the topological connectivity of vehicles through a transmission range, which provides an upper bound to cases that consider factors of channel conflicts, signal fading, etc. These models may serve as a framework for future efforts to consider those other factors.

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